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EQUATIONS OF HIGH FREQUENCY VIBRATIONS OF THERMOPIEZOELECTRIC CRYSTAL PLATES

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Abstract—A system of two-dimensional equations is derived for high frequency motions of crystal plates accounting for coupling of mechanical, electrical and thermal fields.

1. INTRODUCTION

In a recent paper [1], a system of two-dimensional equations, somewhat simpler in form than previous ones, was derived for problems of vibrations of crystal plates in which there is coupling between elastic and electric fields at frequencies as high as those of the fundamental thickness–shear modes. In the present paper, the equations are extended to accommodate coupling with a thermal field. There are already available the plate-equations derived by Tasi and Herrmann [2], which include coupling of elastic and thermal fields, but a different method for treating the thermal field is employed here in order to deal with all three fields in the same way as in [1]. The mechanical displacement, electric potential and temperature change are all expanded in power series of the thickness-coordinate of the plate. The expansions are inserted in an integral energy-equation including a dissipation function instead of in a variational principle with an adjoint field to receive the dissipated energy. The energy-equation is analogous to Biot's [3] variational principle for the equations of thermoelasticity.

Following the substitution of the series expressions in the integral equation of energy balance, the integration with respect to the thickness coordinate is performed and two-dimensional equations of order n, analogous to the three-dimensional equations, are extracted. This is followed by discard of variables and equations of the second and higher orders and the usual adjustments of the remaining members. Finally, a uniqueness theorem, analogous to Weiner's [4] for the thermoelastic case, is devised to establish various face- and edge-conditions sufficient to assure unique solutions of the two-dimensional equations.

2. THREE-DIMENSIONAL, LINEAR THERMOPIEZOELECTRICITY

The equations of the classical, linear theory of thermopiezoelectricity may be grouped as divergence, gradient and constitutive equations for the mechanical, electrical and thermal fields:

Divergence equations

$$T_{ij,i} = \rho \ddot{u}_j, \qquad D_{i,i} = 0, \qquad h_{i,i} = -\Theta_0 \dot{\eta} \tag{1}$$

where T_{ij}, D_i, h_i, u_j, η, ρ and Θ_0 are, respectively, the stress, electric displacement, heat flux, mechanical displacement, entropy density, mass density and reference temperature.

Gradient equations

$$S_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \qquad E_i = -\varphi_{,i}, \qquad h_i = -\kappa_{ij}\theta_{,j} \tag{2}$$

where S_{ij}, E_i, φ, κ_{ij} and θ are, respectively, the strain, electric field, heat conduction coefficient and small temperature change: $|\theta| \ll \Theta_0$.

Constitutive equations

$$T_{ij} = \partial G/\partial S_{ij}, \qquad D_i = -\partial G/\partial E_i, \qquad \eta = -\partial G/\partial\theta, \tag{3}$$

where G is the "electric Gibbs function" ([5], p. 34).

Equations (1), (2) and (3) comprise the twenty-seven equations of linear thermopiezoelectricity governing the twenty-seven dependent variables u_i, T_{ij}, S_{ij}, D_i, E_i, φ, h_i, η, θ. The thermodynamic potential G is related to the potential energy density U by

$$G = U - E_i D_i - \eta\Theta$$

in which Θ, $=\Theta_0 + \theta$, is the absolute temperature. Explicitly, in terms of the variables S_{ij}, E_i, θ and Mason's [5] symbols for the material coefficients,

$$G = \tfrac{1}{2}c^{E\theta}_{ijkl}S_{ij}S_{kl} - \varepsilon^{S\theta}_{ij}E_iE_j - \rho C^E_v\Theta_0^{-1}\theta^2 - e^{\theta}_{ijk}E_iS_{jk} - p^S_i\theta E_i - \lambda^E_{ij}S_{ij}\theta. \tag{4}$$

From (3) and (4), omitting, in the sequel, the superscripts E, S, θ on the symbols for the material coefficients, we find:

$$\begin{aligned} T_{ij} &= c_{ijkl}S_{kl} - e_{kij}E_k - \lambda_{ij}\theta, \\ D_j &= e_{jkl}S_{kl} + \varepsilon_{jk}E_k + p_j\theta, \\ \eta &= \lambda_{kl}S_{kl} + p_kE_k + \alpha\theta \end{aligned} \tag{5}$$

where $\alpha = \rho C^E_v\Theta_0^{-1}$.

By successive substitution, the twenty-seven equations may be reduced to five on u_i, φ and θ:

$$\begin{aligned} c_{ijkl}u_{k,li} + e_{kij}\varphi_{,ki} - \lambda_{ij}\theta_{,i} &= \rho\ddot{u}_j, \\ e_{kij}u_{i,jk} - \varepsilon_{ij}\varphi_{,ij} + p_i\theta_{,i} &= 0, \\ \lambda_{ij}\dot{u}_{i,j} - p_i\dot{\varphi}_{,i} + \alpha\dot{\theta} &= \Theta_0^{-1}\kappa_{ij}\theta_{,ij}. \end{aligned}$$

3. ENERGY EQUATION

In a body occupying a volume V, bounded by a surface S with outward normal $\mathbf{n}$, the principle of conservation of energy, appropriate to (1), (2) and (3), can be expressed [6] in a form analogous to Biot's [3] variational principle for the thermoelastic case:

$$\int_V (\dot{K} + \dot{B} + 2F)\,\mathrm{d}V = \int_S (t_j\dot{u}_j - \dot{\sigma}\varphi - \nu\theta\Theta_0^{-1})\,\mathrm{d}S, \tag{6}$$

where

$$t_j = n_iT_{ij}, \qquad \sigma = n_iD_i, \qquad \nu = n_ih_i, \quad \text{on } S, \tag{7}$$

i.e. t_j is the surface traction, σ is the surface charge and ν is the normal component of the heat flux (across the surface). In the volume integral in (6), K is the kinetic energy density:

$$K = \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i,$$

F is the dissipation function:

$$F = \tfrac{1}{2}\kappa_{ij}\theta_{,i}\theta_{,j}\Theta_0^{-1} = -\tfrac{1}{2}h_i\theta_{,i}\Theta_0^{-1},$$

and B is Biot's generalized free energy density [3]:

$$B = B(S_{ij}, D_i, \eta) = U - \Theta_0\eta = G + E_i D_i + \eta\theta, \tag{8}$$

with

$$T_{ij} = \partial B/\partial S_{ij}, \qquad E_i = \partial B/\partial D_i, \qquad \theta = \partial B/\partial\eta.$$

Now,

$$\begin{aligned}\dot{B} &= (\partial B/\partial S_{ij})\dot{S}_{ij} + (\partial B/\partial D_i)\dot{D}_i + (\partial B/\partial\eta)\dot{\eta},\\ &= T_{ij}\dot{u}_{j,i} - \varphi_{,i}\dot{D}_i + \theta\dot{\eta},\\ &= (T_{ij}\dot{u}_j)_{,i} - T_{ij,i}\dot{u}_j - (\varphi\dot{D}_i)_{,i} + \dot{D}_{i,i}\varphi + \dot{\eta}\theta.\end{aligned}$$

Hence,

$$\int_V \dot{B}\,\mathrm{d}V = \int_S n_i(T_{ij}\dot{u}_j - \varphi\dot{D}_i)\,\mathrm{d}S - \int_V (T_{ij}\dot{u}_j - \dot{D}_{i,i}\varphi - \dot{\eta}\theta)\,\mathrm{d}V. \tag{9}$$

Also,

$$2F = -h_i\theta_{,i}\Theta_0^{-1} = -[(h_i\theta)_{,i} - h_{i,i}\theta]\Theta_0^{-1},$$

so that

$$\int_V 2F\,\mathrm{d}V = -\int_S n_i h_i\theta\Theta_0^{-1}\,\mathrm{d}S + \int_V h_{i,i}\theta\Theta_0^{-1}\,\mathrm{d}V. \tag{10}$$

Finally,

$$\dot{K} = \rho\ddot{u}_i\dot{u}_i. \tag{11}$$

Upon substituting (9), (10) and (11) in (6), we find

$$\int_V [(T_{ij,i} - \rho\ddot{u}_j)\dot{u}_j - \dot{D}_{i,i}\varphi - (\dot{\eta} + h_{i,i}\Theta_0^{-1})\theta]\,\mathrm{d}V$$

$$- \int_S [(n_i T_{ij} - t_j)\dot{u}_j - (n_i\dot{D}_i - \dot{\sigma})\varphi - (n_i h_i - v)\theta\Theta_0^{-1}]\,\mathrm{d}S = 0. \tag{12}$$

It will be observed that the coefficients of $\dot{u}_j$, φ and θ, set equal to zero, give the divergence equations (1) and the boundary values (7). This property will be employed in deriving analogous plate-equations after expanding u_j, φ and θ in power series of the thickness coordinate x_2 and integrating with respect to x_2.

4. UNIQUENESS OF SOLUTIONS

Paralleling Weiner's [4] uniqueness theorem for the thermoelastic case, the reverse of the procedure leading from (6) to (12) can be employed to establish boundary conditions sufficient to assure unique solutions of the twenty-seven equations (1–3).

Consider two sets of the twenty-seven dependent variables u_i, T_{ij}, S_{ij}, D_i, E_i, φ, h_i, η, θ (initially zero) distinguished by prime and double prime, each set comprising a solution of the twenty-seven equations. Let

$$u_i^* = u_i' - u_i'', \qquad T_{ij}^* = T_{ij}' - T_{ij}'', \ \text{etc.}$$

Since the twenty-seven equations are linear, the difference variables, u_i^* etc. are also solutions. We write the equation

$$\int_V [(T_{ij,i}^* - \rho \ddot{u}_j^*)\dot{u}_j^* - \dot{D}_{i,i}^*\varphi^* - (\dot{\eta}^* + h_{i,i}^*\Theta_0^{-1})\theta^*]\,\mathrm{d}V = 0 \tag{13}$$

which is satisfied in view of (1). By the chain rule of differentiation and the divergence theorem (omitting the stars, temporarily):

$$\int_V T_{ij,i}\dot{u}_j\,\mathrm{d}V = \int_V [(T_{ij}\dot{u}_j)_{,i} - T_{ij}\dot{u}_{j,i}]\,\mathrm{d}V = \int_S n_i T_{ij}\dot{u}_j\,\mathrm{d}S - \int_V T_{ij}\dot{u}_{j,i}\,\mathrm{d}V,$$

$$\int_V \dot{D}_{i,i}\varphi\,\mathrm{d}V = \int_V [(\dot{D}_i\varphi)_{,i} - \dot{D}_i\varphi_{,i}]\,\mathrm{d}V = \int_S n_i\dot{D}_i\varphi\,\mathrm{d}S - \int_V \dot{D}_i\varphi_{,i}\,\mathrm{d}V,$$

$$\int_V h_{i,i}\theta\,\mathrm{d}V = \int_V [(h_i\theta)_{,i} - h_i\theta_{,i}]\,\mathrm{d}V = \int_S n_i h_i\theta\,\mathrm{d}S - \int_V h_i\theta_{,i}\,\mathrm{d}V.$$

Employing (2), we have

$$\int_V T_{ij}\dot{u}_{j,i}\,\mathrm{d}V = \int_V T_{ij}\dot{S}_{ij}\,\mathrm{d}V,$$

$$\int_V \dot{D}_i\varphi_{,i}\,\mathrm{d}V = -\int_V \dot{D}_i E_i\,\mathrm{d}V,$$

$$\int_V h_i\theta_{,i}\,\mathrm{d}V = -\int_V \kappa_{ij}\theta_{,i}\theta_{,j}\,\mathrm{d}V = -\Theta_0\int_V 2F\,\mathrm{d}V.$$

Now,

$$T_{ij}\dot{S}_{ij} + \dot{D}_i E_i + \dot{\eta}\theta = T_{ij}\dot{S}_{ij} - D_i\dot{E}_i - \eta\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}(E_i D_i + \eta\theta).$$

This becomes, with (3) (the last of the twenty-seven equations)

$$T_{ij}\dot{S}_{ij} + \dot{D}_i E_i + \dot{\eta}\theta = \frac{\partial G}{\partial S_{ij}}\dot{S}_{ij} + \frac{\partial G}{\partial E_i}\dot{E}_i + \frac{\partial G}{\partial \theta}\dot{\theta} + \frac{\mathrm{d}}{\mathrm{d}t}(E_i D_i + \eta\theta)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}(G + E_i D_i + \eta\theta) = \frac{\mathrm{d}B}{\mathrm{d}t}.$$

Finally,

$$\rho\ddot{u}_i\dot{u}_i = \mathrm{d}K/\mathrm{d}t.$$

Assembling these results and inserting them in (18), we have, with the stars reinstated,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (K^* + B^*)\,\mathrm{d}V = \int_S n_i(T_{ij}^*\dot{u}_j^* - \dot{D}_i^*\varphi^* - h_i^*\theta^*\Theta_0^{-1})\,\mathrm{d}S - \int_V 2F^*\,\mathrm{d}V. \tag{14}$$

Now, the individual K, B and F are positive definite, by definition, and initially zero; so that K^*, B^* and F^*, calculated from the difference variables, have the same properties.

Hence, if the integrand of the surface integral in (14) vanishes, K^* and B^* must be zero and that is possible only if the two solutions are identical. Conditions sufficient to make the integrand of the surface integral vanish (and, hence, conditions sufficient for a unique solution) are the specification, at each point of S, of: one member of each of the three products of traction and displacement components (referred to orthogonal directions α, β, γ at the point) $T_{\alpha\alpha}u_\alpha$, $T_{\alpha\beta}u_\beta$, $T_{\alpha\gamma}u_\gamma$; *and* either the surface charge, $n_i D_i$, or the surface potential, φ; *and* either the heat flux across the surface, $n_i h_i$, or the surface temperature change, θ.

Various conditions arising from interactions with the exterior of V may also be specified. Of special interest, here, is the radiation condition:

$$n_i h_i - k\theta = 0, \quad \text{on } S, \tag{15}$$

where k is a positive constant ranging from zero for an adiabatic boundary to infinity (i.e. $\theta = 0$) for an isothermal boundary. Suppose suitable boundary conditions have been specified for the mechanical and electrical fields so that (14) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (K^* + B^*)\,\mathrm{d}V = -\Theta_0^{-1}\int_S n_i h_i^* \theta^*\,\mathrm{d}S - \int_V 2F^*\,\mathrm{d}V. \tag{16}$$

Add $\Theta_0^{-1}\int_S k\theta^*\theta^*\,\mathrm{d}S$ to each side of (16):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V (K^* + B^*)\,\mathrm{d}V + \Theta_0^{-1}\int_S K\theta^*\theta^*\,\mathrm{d}S = -\Theta_0^{-1}\int_S (n_i h_i^* - k\theta^*)\theta^*\,\mathrm{d}S - \int_V 2F^*\,\mathrm{d}V.$$

Then, since $k\theta^*\theta^*\Theta_0^{-1}$ is positive, the same argument as before yields (15) as an acceptable boundary condition for the thermal field.

5. SERIES OF TWO-DIMENSIONAL EQUATIONS

The plate is referred to rectangular coordinates x_i with the faces, of area A, at $x_2 = \pm b$ and with x_1 and x_3 the coordinates in the middle plane which intersects the right cylindrical or prismatic boundary of the plate in a curve or polygon C. We assume that u_i, φ and θ can be approximated by a power series in x_2:

$$u_i = \sum_n x_2^n u_i^{(n)}, \qquad \varphi = \sum_n x_2^n \varphi^{(n)}, \qquad \theta = \sum_n x_2^n \theta^{(n)}, \tag{17}$$

where $u_i^{(n)}$, $\varphi^{(n)}$ and $\theta^{(n)}$, $n = 0, 1, 2 \ldots$ are functions of x_1, x_3 and t only. These expansions are to be substituted in the energy-balance equation (12) and the integrations with respect to x_2 performed. By the same procedure as in [1], the volume integral in (12) becomes

$$\begin{aligned}\sum_n \int_A \Big(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + [x_2^n T_{2j}]_{-b}^{b} - \rho \sum_m B_{mn}\ddot{u}_j^{(m)}\Big)\dot{u}_j^{(n)}\,\mathrm{d}A \\ - \sum_n \int_A (\dot{D}_{i,i}^{(n)} - n\dot{D}_2^{(n-1)} + [x_2^n \dot{D}_2]_{-b}^{b})\varphi^{(n)}\,\mathrm{d}A \\ - \sum_n \int_A (H_{i,i}^{(n)} - nH_2^{(n-1)} + [x_2^n h_2]_{-b}^{b} + \Theta_0\dot{\eta}^{(n)})\Theta_0^{-1}\theta^{(n)}\,\mathrm{d}A,\end{aligned} \tag{18}$$

where

$$(T_{ij}^{(n)}, D_i^{(n)}, \eta^{(n)}, H_i^{(n)}) = \int_{-b}^{b} x_2^n (T_{ij}, D_i, \eta, h_i)\,\mathrm{d}x_2, \tag{19}$$

$$B_{mn} = 2b^{m+n+1}/m + n + 1, \quad m + n \text{ even}$$

and $B_{mn} = 0$ for $m + n$ odd. Also, as in [1], the surface integral in (12) becomes the sum of area integrals over the two faces of the plate and a line integral around the boundary C:

$$-\sum_n \int_A [(T_{2j} - t_j)x_2^n \dot{u}_j^{(n)} - (\dot{D}_2 - \dot{\sigma})x_2^n \varphi^{(n)} - (h_2 - v)x_2^n \theta^{(n)}\Theta_0^{-1}]_{-b}^{b}\, dA$$

$$-\sum_n \oint_C \int_{-b}^{b} [(n_a T_{aj} - t_j)x_2^n \dot{u}_j^{(n)} - (n_a \dot{D}_a - \dot{\sigma})x_2^n \varphi^{(n)} - (n_a h_a - v)x_2^n \theta^{(n)}\Theta_0^{-1}]_C\, dx_2\, ds, \quad (20)$$

where the index a is planar. Define

$$(F_j^{(n)}, S^{(n)}, N^{(n)}) = [x_2^n(t_j, \sigma, v)]_{-b}^{b}$$

$$(t_j^{(n)}, \sigma^{(n)}, v^{(n)}) = \int_{-b}^{b} x_2^n(t_j, \sigma, v)\, dx_2$$

and insert these in (20). Then set the sum of (18) and (20) equal to zero to obtain the two-dimensional version of (12):

$$\sum_n \int_A (T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} - \rho \sum_m B_{mn} \ddot{u}_j^{(m)})\dot{u}_j^{(n)}\, dA$$

$$-\sum_n \int_A (\dot{D}_{i,i}^{(n)} - n\dot{D}_2^{(n-1)} + S^{(n)})\varphi^{(n)}\, dA$$

$$-\sum_n \int_A (H_{i,i}^{(n)} - nH_2^{(n-1)} + N^{(n)} + \Theta_0 \dot{\eta}^{(n)})\theta^{(n)}\Theta_0^{-1}\, dA$$

$$-\sum_n \oint_C [(n_a T_{aj} - t_j^{(n)})\dot{u}_j^{(n)} - (n_a \dot{D}_a^{(n)} - \dot{\sigma}^{(n)})\varphi^{(n)} - (n_a H_a^{(n)} - v^{(n)})\theta^{(n)}\Theta_0^{-1}]\, ds = 0. \quad (21)$$

Following the argument at the end of Article 3, we extract, from (21), the divergence equations of order n:

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \rho \sum_m B_{mn} \ddot{u}_j^{(m)},$$

$$D_{i,i}^{(n)} - nD_2^{(n-1)} + S^{(n)} = 0,$$

$$H_{i,i}^{(n)} - nH_2^{(n-1)} + N^{(n)} = -\Theta_0 \dot{\eta}^{(n)}$$

and the edge-values of order n:

$$n_a T_{aj}^{(n)} = t_j^{(n)}, \qquad n_a D_a^{(n)} = \sigma^{(n)}, \qquad n_a H_a^{(n)} = v^{(n)} \quad \text{on } C.$$

Proceeding, now, to the gradient equations, we substitute (17) in (2) to obtain

$$S_{ij} = \sum_n x_2^n S_{ij}^{(n)}, \qquad E_i = \sum_n x_2^n E_i^{(n)}, \qquad h_i = \sum_n x_2^n h_i^{(n)}, \quad (22)$$

where $S_{ij}^{(n)}$, $E_i^{(n)}$ and $h_i^{(n)}$ are given by the gradient equations of order n:

$$S_{ij}^{(n)} = \tfrac{1}{2}[u_{j,i}^{(n)} + u_{i,j}^{(n)} + (n+1)(\delta_{i2} u_j^{(n+1)} + \delta_{2j} u_i^{(n+1)})],$$

$$E_i^{(n)} = -\varphi_{,i}^{(n)} - (n+1)\delta_{2i}\varphi^{(n+1)}, \quad (23)$$

$$h_i^{(n)} = -\kappa_{ij}[\theta_{,j}^{(n)} + (n+1)\delta_{2j}\theta^{(n+1)}].$$

For the constitutive equations of order n we find, from (19) and (5):

$$T_{ij}^{(n)} = \sum_m B_{mn}(c_{ijkl} S_{kl}^{(m)} - e_{kij} E_k^{(m)} - \lambda_{ij} \theta^{(m)}),$$
$$D_j^{(n)} = \sum_m B_{mn}(e_{jkl} S_{kl}^{(m)} + \varepsilon_{kj} E_k^{(m)} + p_j \theta^{(m)}), \quad (24)$$
$$\eta^{(n)} = \sum_m B_{mn}(\lambda_{kl} S_{kl}^{(m)} + p_k E_k^{(m)} + \alpha\theta^{(m)}),$$

Finally, from (19) and the third of (22),

$$H_i^{(n)} = -\sum_m B_{mn} \kappa_{ij}[\theta_{,j}^{(m)} + (m+1)\delta_{2j}\theta^{(m+1)}], \quad (25)$$

which, when restricted to orders zero and one, is an alternative to the third of (23): the thermal gradient equation of order n.

This completes the establishment of the two-dimensional, thermopiezoelectric equations of order n.

6. TRUNCATION OF SERIES AND ADJUSTMENT

For frequencies up to those of the fundamental thickness–shear modes, the process of truncation of the series of two-dimensional equations and adjustment of the remaining terms begins with the discard of all the second and higher order terms. Thus, the first of (24) becomes

$$T_{ij}^{(0)} = 2b(c_{ijkl} S_{kl}^{(0)} - e_{kij} E_k^{(0)} - \lambda_{ij} \theta^{(0)}),$$
$$T_{ij}^{(1)} = \tfrac{2}{3}b^3(c_{ijkl} S_{kl}^{(1)} - e_{kij} E_k^{(1)} - \lambda_{ij} \theta^{(1)}) \quad (26)$$

and (25) becomes

$$H_i^{(0)} = -2b\kappa_{ij}(\theta_{,j}^{(0)} + \delta_{2j}\theta^{(1)}), \qquad H_i = -\tfrac{2}{3}b^3\kappa_{ij}\theta_{,j}^{(1)}, \quad (27)$$

but these are only provisional as adjustments of the type employed by Cauchy [7], Poisson [8], Bresse [9] and Timoshenko [10] are to be made in order to compensate, in part, for the omission of the higher order terms.

First of all, free development of the thickness-stretch strain $S_{22}^{(0)}$, $=u_2^{(1)}$, is allowed by setting $T_{22}^{(0)} = 0$ in (26). The resulting equation is solved for $S_{22}^{(0)}$ and that expression is used to eliminate $S_{22}^{(0)}$ from the remaining $T_{ij}^{(0)}$, yielding

$$T_{ij}^{(0)} = 2b(\bar{c}_{ijkl} S_{kl}^{(0)} - \bar{e}_{kij} E_k^{(0)} - \bar{\lambda}_{ij} \theta^{(0)}),$$

where

$$\bar{c}_{ijkl} = c_{ijkl} - c_{ij22} c_{22kl}/c_{2222}, \qquad \bar{e}_{kij} = e_{kij} - c_{ij22} e_{k22}/c_{2222},$$
$$\bar{\lambda}_{ij} = \lambda_{ij} - c_{ij22} \lambda_{22}/c_{2222}.$$

Then the Bresse–Timoshenko shear-correction factors, κ_1 and κ_3, are introduced by the replacement of $\bar{c}$, $\bar{e}$ and $\bar{\lambda}$ by

$$c_{ijkl}^{(0)} = \kappa_{i+j-2}^{\mu} \kappa_{k+l-2}^{\nu} \bar{c}_{ijkl}, \qquad e_{kij}^{(0)} = \kappa_{i+j-2}^{\mu} \bar{e}_{kij}, \qquad \lambda_{ij}^{(0)} = \kappa_{i+j-2}^{\mu} \bar{\lambda}_{ij} \quad \text{(not summed)},$$

where μ and ν are the powers $\cos^2(ij\pi/2)$ and $\cos^2(kl\pi/2)$, respectively. Thus the final expression for $T_{ij}^{(0)}$ is

$$T_{ij}^{(0)} = 2b(c_{ijkl}^{(0)} S_{kl}^{(0)} - e_{kij}^{(0)} E_k^{(0)} - \lambda_{ij}^{(0)} \theta^{(0)}). \quad (28)$$

In the case of the first order strains, free development of all three $S_{2j}^{(1)}$ is required and effected by setting $T_{2j}^{(1)} = 0$ in (26) and using the resulting three equations to eliminate the $S_{2j}^{(1)}$ from the remaining $T_{ij}^{(1)}$ with the result:

$$T_{ab}^{(1)} = \tfrac{2}{3}b^3(c_{abcd}^{(1)} S_{cd}^{(1)} - e_{cab}^{(1)} E_c^{(1)} - \lambda_{ab}^{(1)}\theta^{(1)}), \tag{29}$$

in which the indices a, b, c ... range over 1 and 3 only; the $c_{abcd}^{(1)}$ are the Voigt constants γ [11] and

$$e_{cab}^{(1)} = e_{cij} s_{ijde} c_{abde}^{(1)}, \qquad \lambda_{ab}^{(1)} = \lambda_{ij} s_{ijcd} c_{abcd}^{(1)},$$

where the s_{ijkl} are the elastic compliances.

It will be observed that $u_2^{(1)}$ no longer appears in (28) and (29) and that displacement component is eliminated entirely by dropping it from the kinetic energy density.

Further, following Tasi and Herrmann [2], thermal correction factors κ_1^T and κ_3^T may be introduced by replacing the heat conduction coefficients κ_{ab}, in the expression for $H_a^{(1)}$, from (27), by

$$\kappa_{ab} \to \kappa_{ab}\kappa_a^T\kappa_b^T \quad \text{(not summed)}.$$

An "electric Gibbs function" $\bar{G}$ that produces (28 and 29) through

$$T_{ij}^{(0)} = \partial\bar{G}/\partial S_{ij}^{(0)}, \qquad T_{ab}^{(1)} = \partial\bar{G}/\partial S_{ab}^{(1)} \tag{30}$$

is

$$\begin{aligned}\bar{G} = b(c_{ijkl}^{(0)} S_{ij}^{(0)} S_{kl}^{(0)} - \varepsilon_{ij} E_i^{(0)} E_j^{(0)} - \alpha\theta^{(0)}\theta^{(0)} - 2e_{ijk}^{(0)} E_i^{(0)} S_{jk}^{(0)} - 2p_i\theta^{(0)} E_i^{(0)} - 2\lambda_{ij}^{(0)} S_{ij}^{(0)}\theta^{(0)}) \\ + \tfrac{1}{3}b^3(c_{abcd}^{(1)} S_{ab}^{(1)} S_{cd}^{(1)} - \varepsilon_{ab} E_a^{(1)} E_b^{(1)} - \alpha\theta^{(1)}\theta^{(1)} - 2e_{abc}^{(1)} E_a^{(1)} S_{bc}^{(1)} \\ - 2p_a\theta^{(1)} E_a^{(1)} - 2\lambda_{ab}^{(1)} S_{ab}^{(1)}\theta^{(1)}).\end{aligned} \tag{31}$$

Finally, we define $D_i^{(0)}$, $D_a^{(1)}$, $\eta^{(0)}$ and $\eta^{(1)}$ by

$$D_i^{(0)} = -\partial\bar{G}/\partial E_i^{(0)}, \quad D_a^{(1)} = -\partial\bar{G}/\partial E_a^{(1)}, \quad \eta^{(0)} = -\partial\bar{G}/\partial\theta^{(0)}, \quad \eta^{(1)} = -\partial\bar{G}/\partial\theta^{(1)} \tag{32}$$

to serve as adjusted expressions to replace those obtained from (24).

7. RECAPITULATION

Divergence equations

$$\begin{aligned} &T_{aj,a}^{(0)} + F_j^{(0)} = 2b\rho\ddot{u}_j^{(0)}, && T_{ab,a}^{(1)} - T_{2b}^{(0)} + F_b^{(1)} = \tfrac{2}{3}b^3\rho\ddot{u}_b^{(1)}, \\ &D_{a,a}^{(0)} + S^{(0)} = 0, && D_{a,a}^{(1)} - D_2^{(0)} + S^{(1)} = 0, \\ &H_{a,a}^{(0)} + N^{(0)} = -\Theta_0\dot{\eta}^{(0)}, && H_{a,a}^{(1)} - H_2^{(0)} + N^{(1)} = -\Theta_0\dot{\eta}^{(1)}; \end{aligned} \tag{33}$$

Gradient equations

$$\begin{aligned} &S_{ij}^{(0)} = \tfrac{1}{2}[u_{j,i}^{(0)} + u_{i,j}^{(0)} + \delta_{i2}u_j^{(1)} + \delta_{2j}u_i^{(1)}], && S_{ab}^{(1)} = \tfrac{1}{2}(u_{b,a}^{(1)} + u_{a,b}^{(1)}), \\ &E_i^{(0)} = -(\varphi_{,i}^{(0)} + \delta_{i2}\varphi^{(1)}), && E_a^{(1)} = -\varphi_{,a}^{(1)}, \\ &H_i^{(0)} = -2b\kappa_{ij}(\theta_{,j}^{(0)} + \delta_{2j}\theta^{(1)}), && H_a^{(1)} = -\tfrac{2}{3}b^3\kappa_{ab}^{(1)}\theta_{,b}^{(1)}; \end{aligned} \tag{34}$$

Constitutive equations

$$
\begin{aligned}
T_{ij}^{(0)} &= 2b(c_{ijkl}^{(0)}S_{kl}^{(0)} - e_{kij}^{(0)}E_k^{(0)} - \lambda_{ij}^{(0)}\theta^{(0)}),\\
T_{ab}^{(1)} &= \tfrac{2}{3}b^3(c_{abcd}^{(1)}S_{cd}^{(1)} - e_{cab}^{(1)}E_c^{(1)} - \lambda_{ab}^{(1)}\theta^{(1)}),\\
D_j^{(0)} &= 2b(e_{jkl}^{(0)}S_{kl}^{(0)} + \varepsilon_{jk}E_k^{(0)} + p_j\theta^{(0)}),\\
D_b^{(1)} &= \tfrac{2}{3}b^3(e_{bcd}^{(1)}S_{cd}^{(1)} + \varepsilon_{bc}E_c^{(1)} + p_b\theta^{(1)}),\\
\eta^{(0)} &= 2b(\lambda_{kl}^{(0)}S_{kl} + p_kE_k^{(0)} + \alpha\theta^{(0)}),\\
\eta^{(1)} &= \tfrac{2}{3}b^3(\lambda_{cd}^{(1)}S_{cd}^{(1)} + p_cE_c^{(1)} + \alpha\theta^{(1)}),
\end{aligned}
\tag{35a}
$$

or

$$
T_{ij}^{(0)} = \frac{\partial\bar{G}}{\partial S_{ij}^{(0)}},\qquad T_{ab}^{(1)} = \frac{\partial\bar{G}}{\partial S_{ab}^{(1)}},\qquad D_i^{(0)} = -\frac{\partial\bar{G}}{\partial E_i^{(0)}},\qquad D_a^{(1)} = -\frac{\partial\bar{G}}{\partial E_a^{(1)}},
$$
$$
\eta^{(0)} = -\frac{\partial\bar{G}}{\partial\theta^{(0)}},\qquad \eta^{(1)} = -\frac{\partial\bar{G}}{\partial\theta^{(1)}},
\tag{35b}
$$

where $\bar{G}$ is given by (31);

Equations† *on* $u_i^{(0)}$, $u_a^{(1)}$, $\varphi^{(0)}$, $\varphi^{(1)}$, $\theta^{(0)}$, $\theta^{(1)}$

$$
\begin{aligned}
&c_{ijkl}^{(0)}(u_{k,li}^{(0)} + \delta_{2l}u_{k,i}^{(1)}) + e_{kij}^{(0)}(\varphi_{,ki}^{(0)} + \delta_{2k}\varphi_{,i}^{(1)}) - \lambda_{ij}^{(0)}\theta_{,i}^{(0)} + \tfrac{1}{2}b^{-1}F_j^{(0)} = \rho\ddot{u}_j^{(0)},\\
&\quad e_{kij}^{(0)}(u_{j,ki}^{(0)} + \delta_{2i}u_{j,k}^{(1)}) - \varepsilon_{ij}(\varphi_{,ij}^{(0)} + \delta_{i2}\varphi_{,i}^{(1)}) + p_i\theta_{,i}^{(0)} + \tfrac{1}{2}b^{-1}S^{(0)} = 0,\\
&\lambda_{ij}^{(0)}(\dot{u}_{j,i}^{(0)} + \delta_{2i}\dot{u}_j^{(1)}) - p_i(\dot{\varphi}_{,i}^{(0)} + \delta_{i2}\dot{\varphi}^{(1)}) + \alpha\dot{\theta}^{(0)} + \tfrac{1}{2}b^{-1}\Theta_0^{-1}N^{(0)} = \Theta_0^{-1}\kappa_{ij}(\theta_{,ij}^{(0)} + \delta_{2j}\theta_{,i}^{(1)})
\end{aligned}
\tag{36}
$$

$$
\begin{aligned}
&c_{abcd}^{(1)}u_{d,ca}^{(1)} + e_{cab}^{(1)}\varphi_{,ca}^{(1)} - \lambda_{ab}^{(1)}\theta_{,a}^{(1)} - 3b^{-2}[c_{2bkl}^{(0)}(u_{l,k}^{(0)} + \delta_{2k}u_l^{(1)})\\
&\qquad + e_{i2b}^{(0)}(\varphi_{,i}^{(0)} + \delta_{i2}\varphi^{(1)}) - \lambda_{2b}^{(0)}\theta^{(0)}] + \tfrac{3}{2}b^{-3}F_b^{(1)} = \rho\ddot{u}_b^{(1)},\\
&e_{cab}^{(1)}u_{b,ca}^{(1)} - \varepsilon_{bc}\varphi_{,bc}^{(1)} + p_a\theta_{,a}^{(1)} - 3b^{-2}[e_{2ij}^{(0)}(u_{j,i}^{(0)} + \delta_{2i}u_j^{(1)})\\
&\qquad - \varepsilon_{i2}(\varphi_{,i}^{(0)} + \delta_{i2}\varphi^{(1)}) + p_2\theta^{(0)}] + \tfrac{3}{2}b^{-3}S^{(1)} = 0,\\
&\lambda_{ab}^{(1)}\dot{u}_{b,a}^{(1)} - p_a\dot{\varphi}_{,a}^{(1)} + \alpha\dot{\theta}_a^{(1)} - 3b^{-2}\kappa_{2j}(\theta_{,j}^{(0)} + \delta_{2j}\theta^{(1)}) + \tfrac{3}{2}b^{-3}\Theta_0^{-1}N^{(1)} = \Theta_0^{-1}\kappa_{ab}^{(1)}\theta_{,ab}^{(1)}.
\end{aligned}
\tag{37}
$$

In addition, it is convenient to define a kinetic energy density and a dissipation function:

$$
\bar{K} = \rho b\dot{u}_j^{(0)}\dot{u}_j^{(0)} + \tfrac{1}{3}\rho b^3\dot{u}_b^{(1)}\dot{u}_b^{(1)},
\tag{38}
$$

$$
\bar{F} = b\Theta_0^{-1}\kappa_{ij}(\theta_{,i}^{(0)} + \delta_{i2}\theta^{(1)})(\theta_{,j}^{(0)} + \delta_{2j}\theta^{(1)}) + \tfrac{1}{3}b^3\Theta_0^{-1}\kappa_{ab}^{(1)}\theta_{,a}^{(1)}\theta_{,b}^{(1)}.
\tag{39}
$$

8. UNIQUENESS AND FACE- AND EDGE-CONDITIONS

The plate equations comprise the forty-two equations (33–35) (a or b) on the forty-two dependent variables $u_i^{(0)}, u_a^{(1)}, T_{ij}^{(0)}, T_{ab}^{(1)}, S_{ij}^{(0)}, S_{ab}^{(1)}, D_i^{(0)}, D_a^{(1)}, E_i^{(0)}, E_a^{(1)}, \varphi^{(0)}, \varphi^{(1)}, H_i^{(0)}, H_a^{(1)}$, $\eta^{(0)}, \eta^{(1)}, \theta^{(0)}, \theta^{(1)}$. As before, we consider two sets of solutions, initially zero, identified by prime and double prime, and their differences

$$
u_i^{(0)*} = u_i^{(0)\prime} - u_i^{(0)\prime\prime},\quad \text{etc.}
$$

† Note that the first two of (36) contain terms in $\varphi^{(1)}$ which were inadvertently omitted in the corresponding equations in [1].

which are also solutions owing to the linearity of the equations. We form the equation

$$\int_A [(T^{(0)*}_{aj,a} + F^{(0)*}_j - 2b\rho\ddot{u}^{(0)*}_j)\dot{u}^{(0)*}_j + (T^{(1)*}_{ab,a} - T^{(0)*}_{2b} + F^{(1)*}_b - \tfrac{2}{3}b^3\rho\ddot{u}^{(1)*}_b)\dot{u}^{(1)*}_b]\,\mathrm{d}A$$

$$- \int_A [(\dot{D}^{(0)*}_{a,a} + \dot{S}^{(0)*})\varphi^{(0)*} + (\dot{D}^{(1)*}_{a,a} - \dot{D}^{(0)*}_2 + \dot{S}^{(1)*})\varphi^{(1)*}]\,\mathrm{d}A$$

$$- \int_A \Theta_0^{-1}[(H^{(0)*}_{a,a} + N^{(0)*} + \Theta_0\dot{\eta}^{(0)*})\theta^{(0)*}$$

$$+ (H^{(1)*}_{a,a} - H^{(0)*}_2 + N^{(1)*} + \Theta_0\dot{\eta}^{(1)*})\theta^{(1)*}]\,\mathrm{d}A = 0, \quad (40)$$

which is satisfied in view of (33). Omitting stars, temporarily, we have

$$\int_A [T^{(0)}_{aj,a}\dot{u}^{(0)}_j + (T^{(1)}_{ab,a} - T^{(0)}_{2b})\dot{u}^{(1)}_b]\,\mathrm{d}A = \oint_C n_a(T^{(0)}_{aj}\dot{u}^{(0)}_j + T^{(1)}_{ab}\dot{u}^{(1)}_b)\,\mathrm{d}s$$

$$- \int_A [T^{(0)}_{ij}(\dot{u}^{(0)}_{j,i} + \delta_{i2}\dot{u}^{(1)}_j) + T^{(1)}_{ab}\dot{u}^{(1)}_{b,a}]\,\mathrm{d}A,$$

$$\int_A [\dot{D}^{(0)}_{a,a}\varphi^{(0)} + (\dot{D}^{(1)}_{a,a} - \dot{D}^{(0)}_2)\varphi^{(1)}]\,\mathrm{d}A = \oint_C n_a(\dot{D}^{(0)}_a\varphi^{(0)} + \dot{D}^{(1)}_a\varphi^{(1)})\,\mathrm{d}s$$

$$- \int_A [\dot{D}^{(0)}_i(\varphi^{(0)}_{,i} + \delta_{i2}\varphi^{(1)}) + \dot{D}^{(1)}_a\varphi^{(1)}_{,a}]\,\mathrm{d}A,$$

$$\int_A [H^{(0)}_{a,a}\theta^{(0)} + (H^{(1)}_{a,a} - H^{(0)}_2)\theta^{(1)}]\,\mathrm{d}A = \oint_C n_a(H^{(0)}_a\theta^{(0)} + H^{(1)}_a\theta^{(1)})\,\mathrm{d}s$$

$$- \int_A [H^{(0)}_i(\theta^{(0)}_{,i} + \delta_{i2}\theta^{(1)}) + H^{(1)}_a\theta^{(1)}_{,a}]\,\mathrm{d}A.$$

Employing (34) we find

$$\int_A [T^{(0)}_{ij}(\dot{u}^{(0)}_{j,i} + \delta_{i2}\dot{u}^{(1)}_j) + T^{(1)}_{ab}u^{(1)}_{b,a}]\,\mathrm{d}A = \int_A (T^{(0)}_{ij}\dot{S}^{(0)}_{ij} + T^{(1)}_{ab}\dot{S}^{(1)}_{ab})\,\mathrm{d}A,$$

$$\int_A [\dot{D}^{(0)}_i(\varphi^{(0)}_{,i} + \delta_{i2}\varphi^{(1)}) + \dot{D}^{(1)}_a\varphi^{(1)}_{,a}]\,\mathrm{d}A = -\int_A (\dot{D}^{(0)}_i E^{(0)}_i + \dot{D}^{(1)}_a E^{(1)}_a)\,\mathrm{d}A,$$

$$\int_A [H^{(0)}_i(\theta^{(0)}_{,i} + \delta_{i2}\theta^{(1)}) + H^{(1)}_a\theta^{(1)}_{,a}]\,\mathrm{d}A = -\int_A [2b\kappa_{ij}(\theta^{(0)}_{,i} + \delta_{i2}\theta^{(1)})(\theta^{(0)}_{,j} + \delta_{2j}\theta^{(1)})$$

$$+ \tfrac{2}{3}b^3\kappa^{(1)}_{ab}\theta^{(1)}_{,a}\theta^{(1)}_{,b}]\,\mathrm{d}A$$

$$= -\Theta_0\int_A 2\bar{F}\,\mathrm{d}A, \quad \text{from (39).}$$

Now,

$$T^{(0)}_{ij}\dot{S}^{(0)}_{ij} + T^{(1)}_{ab}\dot{S}^{(1)}_{ab} + \dot{D}^{(0)}_i E^{(0)}_i + \dot{D}^{(1)}_a E^{(1)}_a + \dot{\eta}^{(0)}\theta^{(0)} + \dot{\eta}^{(1)}\theta^{(1)}$$

$$= T^{(0)}_{ij}\dot{S}^{(0)}_{ij} + T^{(1)}_{ab}\dot{S}^{(1)}_{ab} - D^{(0)}_i\dot{E}^{(0)}_i - D^{(1)}_a\dot{E}^{(1)}_a - \eta^{(0)}\dot{\theta}^{(0)} - \eta^{(1)}\dot{\theta}^{(1)}$$

$$+ \frac{\mathrm{d}}{\mathrm{d}t}(E^{(0)}_i D^{(0)}_i + E^{(1)}_a D^{(1)}_a + \eta^{(0)}\theta^{(0)} + \eta^{(1)}\theta^{(1)})$$

$$= \frac{d\bar{G}}{dt} + \frac{d}{dt}(E_i^{(0)}D_i^{(0)} + E_a^{(1)}D_a^{(1)} + \eta^{(0)}\theta^{(0)} + \eta^{(1)}\theta^{(1)}) \quad \text{from (35b)},$$

$$= d\bar{B}/dt,$$

where

$$\bar{B} = \bar{G} + E_i^{(0)}D_i^{(0)} + E_a^{(1)}D_a^{(1)} + \eta^{(0)}\theta^{(0)} + \eta^{(1)}\theta^{(1)}$$

i.e. by comparison with (8), $\bar{B}$ is the plate analogue of Biot's generalized free energy. Finally, from (38)

$$2b\rho\ddot{u}_j^{(0)}\dot{u}_j^{(0)} + \tfrac{2}{3}b^3\rho\ddot{u}_b^{(1)}\dot{u}_b^{(1)} = d\bar{K}/dt.$$

Assembling all these results and inserting them in (40), the latter becomes

$$\frac{d}{dt}\int_A (\bar{K}^* + \bar{B}^*)\, dA = \int_A A^*\, dA + \oint_C C^*\, ds - \int_A 2\bar{F}^*\, dA, \tag{41}$$

where

$$A^* = F_j^{(0)*}\dot{u}_j^{(0)*} + F_a^{(1)*}\dot{u}_a^{(1)*} - \dot{S}^{(0)*}\varphi^{(0)*} - \dot{S}^{(1)*}\varphi^{(1)*} - \Theta_0^{-1}N^{(0)*}\theta^{(0)*} - \Theta_0^{-1}N^{(1)*}\theta^{(1)*},$$

$$C^* = n_a(T_{aj}^{(0)*}\dot{u}_j^{(0)*} + T_{ab}^{(1)*}\dot{u}_b^{(1)*} - \dot{D}_a^{(0)*}\varphi^{(0)*} - \dot{D}_a^{(1)*}\varphi^{(1)*} - \Theta_0^{-1}H_a^{(0)*}\theta^{(0)*} - \Theta_0^{-1}H_a^{(1)*}\theta^{(1)*}).$$

As in the three-dimensional case, the individual $\bar{K}$, $\bar{B}$ and $\bar{F}$ are positive definite by definition and initially zero, so that $\bar{K}^*$, $\bar{B}^*$ and $\bar{F}^*$, calculated from the difference-variables, have the same properties. Hence, if A^* and C^* are zero, $\bar{K}^*$ and $\bar{B}^*$ must be zero and that is possible only if the two solutions are identical. Conditions sufficient to make A^* zero are: at each point of the interior of the plate, specification of one member of each of the five products of displacement and face–traction components (referred to orthogonal directions α, β, x_2):

$$F_\alpha^{(0)}u_\alpha^{(0)},\ F_\beta^{(0)}u_\beta^{(0)},\ F_2^{(0)}u_2^{(0)},\ F_\alpha^{(1)}u_\alpha^{(1)},\ F_\beta^{(1)}u_\beta^{(1)};$$

and one member of each of the two products of electric potential and face-charge:

$$\varphi^{(0)}S^{(0)}, \qquad \varphi^{(1)}S^{(1)};$$

and one member of each of the two products of temperature change and thermal flux:

$$\theta^{(0)}N^{(0)}, \qquad \theta^{(1)}N^{(1)}.$$

Conditions sufficient to make C^* zero are: at each point of the edge of the plate, specification of one member of each of the five products of edge–displacement and edge–traction components (referred to orthogonal directions, n, s, x_2, with n the outward normal to C):

$$T_{nn}^{(0)}u_n^{(0)},\ T_{ns}^{(0)}u_s^{(0)},\ T_{n2}^{(0)}u_2^{(0)},\ T_{nn}^{(1)}u_n^{(1)},\ T_{ns}^{(1)}u_s^{(1)};$$

and one member of each of the two products of edge–potential and edge–charge:

$$\varphi^{(0)}D_n^{(0)}, \qquad \varphi^{(1)}D_n^{(1)};$$

and one member of each of the two products of edge–temperature change and thermal edge–flux:

$$\theta^{(0)}H_n^{(0)}, \qquad \theta^{(1)}H_n^{(1)}.$$

To find thermal radiation conditions for the faces and edges of the plate, add

$$k\Theta_0^{-1}\int_A(\theta^{(0)*}+x_2\theta^{(1)*})^2_{x_2=b}\,dA+k\Theta_0^{-1}\int_A(\theta^{(0)*}+x_2\theta^{(1)*})^2_{x_2=-b}\,dA$$

$$+k\Theta_0^{-1}\oint_C\int_{-b}^{b}(\theta^{(0)*}+x_2\theta^{(1)*})^2\,dx_2\,ds$$

$$=2k\Theta_0^{-1}\int_A(\theta^{(0)*}\theta^{(0)*}+b^2\theta^{(1)*}\theta^{(1)*})\,dA+2bk\Theta_0^{-1}\oint_C(\theta^{(0)*}\theta^{(0)*}+\tfrac{1}{3}b^2\theta^{(1)*}\theta^{(1)*})\,ds$$

to each side of (41) and note that the additional integrals are positive. The thermal terms on the right-hand side of (41) are then

$$-\Theta_0^{-1}\int_A[(N^{(0)*}-2k\theta^{(0)*})\theta^{(0)*}+(N^{(1)*}-2kb^2\theta^{(1)*})\theta^{(1)*}]\,dA$$

$$-\Theta_0^{-1}\oint_C[(n_aH_a^{(0)*}-2bk\theta^{(0)*})\theta^{(0)*}+(n_aH_a^{(1)*}-\tfrac{2}{3}b^3\theta^{(1)*})\theta^{(1)*}]\,ds.$$

Hence, by the usual argument, the thermal radiation conditions are

$$N^{(0)}-2k\theta^{(0)}=0,\qquad N^{(1)}-2kb^2\theta^{(1)}=0$$

for the faces of the plate and

$$n_aH_a^{(0)}-2bk\theta^{(0)}=0,\qquad n_aH_a^{(1)}-\tfrac{2}{3}b^3k\theta^{(1)}=0$$

for the edge of the plate. The constant k can be different for the interior and edge of the plate as well as different for the two faces of the plate. In the latter case, the fluxes across the two faces have to be distinguished:

$$N^{(0)}=N_+^{(0)}-N_-^{(0)},\qquad N^{(1)}=N_+^{(1)}-N_-^{(1)}$$

where

$$N_\pm^{(0)}=[h_2]_{x_2=\pm b},\qquad N_\pm^{(1)}=[x_2h_2]_{x_2=\pm b}.$$

Then, for the faces of the plate, the radiation conditions are

$$N_+^{(0)}-k_+(\theta^{(0)}+b\theta^{(1)})=0,\qquad N_+^{(1)}-k_+b(\theta^{(0)}+b\theta^{(1)})=0,$$

$$N_-^{(0)}+k_-(\theta^{(0)}-b\theta^{(1)})=0,\qquad N_-^{(1)}-k_-b(\theta^{(0)}-b\theta^{(1)})=0,$$

where k_+ and k_- are the values of k for the faces $x_2=b$ and $x_2=-b$, respectively.

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Абстракт—Разработана двухмерная система уравнений для высокочастотных движений кристаллических пластинок, принимающая во внимание взаимодействие механических, электростатических и термических полей.